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Brief communication

Bianchi identities for an $N = 2$, $d = 5$ supersymmetric Yang–Mills theory on the group manifold

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Abstract

The purpose of this note is the construction of a geometrical structure for a supersymmetric $N = 2$, $d = 5$ Yang–Mills theory on the group manifold. From a general hypothesis proposed for the curvatures of the theory, the Bianchi identities are solved, whose solution will be fundamental for the construction of the geometrical action for the $N = 2$, $d = 5$ supergravity and Yang–Mills coupled theory.

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1. Introduction

The study and the description of the influence of spin on gravitational phenomena is perhaps the main physical motivation of the Einstein–Cartan theory of space–time [1]. The essential idea that underlies this theory has its roots, however, on the mathematical theory of connections which took off in 1916, starting with a paper by the mathematician Hessenberg [2].

A first attempt in formulating a gauge structure for the theory of gravitation had been accomplished by Cartan, already in the twenties, in a series of articles published in "Annales de l'Escole Normale Supérieure" entitled "Sur les variétes à conexxions affines et la théorie de la relativité généralisée" [3]. This is the purpose and the starting point for other modern

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geometrical attempts for extending gravity and supergravity theories [4,5]. Cartan's works remained unknown to most physicists, till Kibble [6], in the sixties, proposed a formulation equivalent to that of Cartan, in his attempt of treating Einstein's gravitation as a Yang–Mills theory. Cartan's ideas reappeared in the important work by Hehl et al. [7] and by Trautman [8]. The review paper by Hehl associates the tensor of torsion to the intrinsic angular momentum, whereas Ref. [8] contains an exposition based in Cartan's differential forms. In recent years, contributions made by Regge and collaborators [4] in Turim, popularly called "Group Manifold approach to gravity and supergravity theories", may be regarded as a differential geometry approach rooted in the Cartan formulation of gravity. In this programme the geometrical scenario is not that of an embedding of a Riemannian manifold into a pseudo-euclidean space, such as the known local embedding of an n-dimensional Riemannian manifold in a $\frac{1}{2}n(n+1)$-dimensional pseudo-euclidean space. The base manifold M is a non-Riemannian manifold (having torsion in general) lying in the curved group manifold. Consequently this approach is formulated in the language of differential forms combined with the exterior product, which are more compact than the usual tensor calculus with an equivalent physical content [9,10].

Motivated by the Turim approach philosophy a geometric structure for an extended five-dimensional gravity theory of Yang–Mills type is presented in Section 2.

From the criteria established throughout this paper, a general hypothesis has been set up for the curvature of the theory and then the Bianchi identities are solved, whose solution will be indispensable for the construction of the geometrical action.

2. Geometrical five-dimensional Yang–Mills theory: supergroup, curvatures and Bianchi identities

The first step of the theory is the choice of the group on which the theory is to be formulated. The group, in its turn, is determined from the $d = 5$ supergravity, since the geometrical SYM (Supersymmetric Yang–Mills) $N = 2, d = 5$ can immediately be coupled to the $N = 2, d = 5$ supergravity. In the case of the supergravity, one can deduce the nature of the group upon analysis of the field content of the theory. The group is characterized by the dual generations of the 1-forms ϖ^{ab} (SO(1, 4) Lorentz connection with 10 parameters), V^a (vierbein associated to the graviton with five parameters), B (spin-1 field with five parameters) and ψA (gravitini with four parameters); it is therefore a 24-parameter group. The group can then be identified as SU(2, 2/1) or one of its contractions [11].

The group of the $N = 2, d = 5$ SYM turns out to be the direct product between that of the supergravity (which we shall assume in its contracted form) and a general gauge group $\mathcal{G}$:

$$G = \mathcal{G} \otimes \overline{\mathrm{SU}(2, 2/1)}. \tag{1}$$

In the supergravity case it is shown that there exists a gauge invariance of the theory with respect to

$$H = \mathrm{SO}(1, 4) \otimes \mathrm{U}(1). \tag{2}$$

Then, the coupling between EGTYM-5 and supergravity acquires a bundle structure, where

$$H' = \mathcal{G} \otimes \mathrm{SO}(1,4) \otimes \mathrm{U}(1) \tag{3}$$

will be the fibre, and the quotient space

$$G/H', \tag{4}$$

the base space of the principal fibre bundle.

The set of curvatures of $G = \mathcal{G} \otimes \overline{\mathrm{SU}(2,2/1)}$ will then be completed—the supergravity curvatures are already established in [11], by F, curvature associated to the gauge group $\mathcal{G}$ and by the covariant derivatives of λ_A, σ, F_{ab}, being:

$$\begin{aligned} &\lambda_A = \text{spinorial field, related to the Dirac equation,} \\ &\Gamma^m \Lambda_{mA} = 0 \quad (\Lambda_{mA} = D\lambda_A); \end{aligned} \tag{5}$$

$$\sigma = \text{scalar field}; \tag{6}$$

$$F_{ab} = \tfrac{1}{2}(\partial_a A_b - \partial_b A_a) = \partial_{[a} A_{b]}. \tag{7}$$

For the theories on a group manifold an important property must be considered in setting the curvatures: the rheonomy [5,10–12]. It is worthwhile to come back once again to the case of $N = 2, d = 5$ supergravity. For this theory, the requirement of non-triviality implies that the components "in" of the curvature (that is, those components with respect to the basis VV) are related to the components "out" (with respect to the basis ψV or $\psi\psi$). Our theory lies in superspace, that is, the fields depend on X^μ and Θ_A (pseudo-Majorama fermionic coordinates). We may consider the space–time M^5 as a hypersurface embedded in superspace.

The components "in" of the curvature are substantially the derivatives of the pseudo-connection along M^5, the components "out" being those along orthonormal directions. The rheonomy therefore is equivalent to the possibility of developing all the dynamics in space–time, giving the theory a physical meaning. In order that the SYM $N = 2$, $d = 5$, formulated in superspace, also acquires a physical meaning, it will be necessary that the derivatives "out" may be expressed in terms of the derivatives "in". Besides the rheonomy another property of a theory on a group manifold is considered: factorization, which is associated to the use of the restricted basis, V^A and ψ_A [10].

We have at this point the elements sufficient for the formulation of explicit expressions for F and for the covariant derivatives. The following other criteria are obviously indispensable:
- respect of the fermionic or bosonic character of the object under consideration;
- consideration of the degree of the forms;
- Lorentz covariance;
- respect of the character of reality;
- dimensional analysis.

A general hypothesis of the expression for the curvatures is then the following:

$$\begin{aligned} F = dA = {} & F_{ab} V^a \wedge V^b + \mathrm{i}c\overline{\lambda_A} \wedge \Gamma_m \psi_A \wedge V^m + \mathrm{i}d\varepsilon_{AB}\overline{\lambda_A} \wedge \Gamma_m \psi_B \wedge V^m \\ & + \mathrm{i}f\sigma \wedge \overline{\psi_A} \wedge \psi_A, \end{aligned} \tag{8}$$

$$D\lambda_A = \Lambda_{mA} \wedge V^m + \mathrm{i}gF_{ab} \wedge \Sigma^{ab}\psi_A + h\phi_a \wedge \Gamma^a\psi_A + \mathrm{i}tF_{ab} \wedge \varepsilon_{AB}\Sigma^{ab}\psi_B + Z\phi_a \wedge \Gamma^a\varepsilon_{AB}\psi_B, \tag{9}$$

$$D\sigma = \phi_a \wedge V^a + \mathrm{i}k\overline{\lambda_A} \wedge \psi_A + \mathrm{i}l\varepsilon_{AB}\overline{\lambda_A} \wedge \psi_B, \tag{10}$$

$$DF^{ab} = G_m^{ab} \wedge V^m + \mathrm{i}n\overline{\Lambda_A}^{[a} \wedge \Gamma^{b]}\psi_A + \mathrm{i}p\varepsilon_{AB}\overline{\Lambda_A}^{[a} \wedge \Gamma^{b]}\psi_B, \tag{11}$$

where $\phi_a = \partial_a\sigma$ and $G_m^{ab} = \partial_m F^{ab}$.

To find the parameters we shall use the integrability conditions:

$$DF = 0, \tag{12}$$

$$DD\lambda_A = 0, \tag{13}$$

$$DD\sigma = 0, \tag{14}$$

$$DDF_{ab} = 0. \tag{15}$$

Such kind of identities on a manifold are usually called Bianchi identities. We shall adopt this nomenclature for the preceding equations, although the last three have more than one covariant derivative. By solving the Bianchi identities for the system of equations we will find that

$$DDF^{ab} = D(G_m{}^{ab}) \wedge V^m + \tfrac{1}{2}\mathrm{i}G_m{}^{ab} \wedge \overline{\psi}_A \wedge \Gamma^m\psi_A - \mathrm{i}nD\overline{\Lambda}_A{}^{[a} \wedge \Gamma^{b]}\psi_A + \mathrm{i}pD\overline{\Lambda}_A{}^{[a} \wedge \Gamma^{b]}\psi_B = 0, \tag{16}$$

where we have already used the fact that $DV^m = \frac{1}{2}\psi_A \wedge \Gamma^m\psi_A$ and $D\psi_A = 0$ if the curvatures of $\overline{\mathrm{SU}(2, 2/1)}$ vanish.

Upon substitution of the expressions of $DG_m{}^{ab}$ and $D\overline{\Lambda}_A^a$ we obtain

$$\begin{aligned}
&D_nG_m{}^{ab} \wedge V^n \wedge V^m + \mathrm{i}nD_m\overline{\Lambda}_a{}^{[a} \wedge \Gamma^{b]}\psi_A \wedge V^m \\
&+ \mathrm{i}pD_m\overline{\Lambda}_B{}^{[a} \wedge \Gamma^{b]}\psi_B \wedge V^m + \tfrac{1}{2}\mathrm{i}G_m{}^{ab} \wedge \overline{\psi}_A \wedge \Gamma^m\psi_A \\
&+ \mathrm{i}nD^{[a}\Lambda_{mA} \wedge V^m \wedge \Gamma^{b]}\psi_A + ngD^{[a}F_{lm} \wedge \overline{\psi}_A \wedge \Sigma^{lm}\Gamma^{b]}\psi_A \\
&+ inhD^{[a}\phi_l \wedge \psi_A\Gamma^l\Gamma^{b]}\psi_A + nt\varepsilon_{AB}D^{[a}F_{lm} \wedge \psi_B\Sigma^{lm}\Gamma^{b]}\psi_A \\
&+ inz\varepsilon_{AB}D^{[a}\phi_l \wedge \psi_B\Gamma^l\Gamma^{b]}\psi_A + \mathrm{i}p\varepsilon_{AB}D^{[a}\overline{\Lambda}_{mA} \wedge V^m \wedge \Gamma^{b]}\psi_A \\
&+ pg\varepsilon_{AB}D^{[a}F_{lm} \wedge \overline{\psi}_A \wedge \Sigma^{lm}\Gamma^{b]}\psi_B + \mathrm{i}ph\varepsilon_{BC}D^{[a}\phi_l \wedge \overline{\psi}_A \wedge \Gamma^l\Gamma^{b]}\psi_C \\
&+ pt\varepsilon_{AB}\varepsilon_{AC}D^{[a}F_{lm} \wedge \psi_B \wedge \Sigma^{lm}\Gamma^{b]}\psi_C \\
&+ \mathrm{i}pz\varepsilon_{AB}\varepsilon_{AC}D^{[a}\phi_l \wedge \psi_B \wedge \Gamma^l\Gamma^{b]}\psi_C = 0.
\end{aligned} \tag{17}$$

The projection on two fünfbeins leads to

$$D_{[n}D_{m]}F^{ab} = 0. \tag{18}$$

By picking those terms containing $\Sigma\Gamma$ and $\Gamma\Gamma$ and developing them in the basis of Dirac matrices, one finds that

(i) $$D^mF^{ab} - 2(ng + pk)D^{[a}F^{b]m} = 0, \tag{19}$$

which coincides with the homogeneous Maxwell equations

$$G_{[ab/m]} = 0, \tag{20}$$

and one then has $ng + pt = 1$; and

$$\text{(ii)} \qquad \tfrac{1}{2}(pg - nt)\varepsilon^{cdlm[b}D^{a]}F_{cd} + 2(ph - nz)\eta^{n[b}D^{a]}\phi^{l} = 0. \tag{21}$$

In order to satisfy the relationship (ii) without requiring the validity of non-physically acceptable relations between the derivatives of different fields, it is necessary that

$$pg - nt = 0, \tag{22}$$

$$ph - nz = 0. \tag{23}$$

We have therefore obtained the first three relations among the parameteres of the curvatures. By using (9) and working in zero supergravity, one has

$$\begin{aligned} DD\sigma = {} & D_a\phi_b \wedge V^a \wedge V^b + \mathrm{i}kD_a\overline{\lambda}_A \wedge \psi_A \wedge V^a + \mathrm{i}l\varepsilon_{AB}D_a\overline{\lambda}_A \wedge \psi_B \wedge V^a \\ & + \tfrac{1}{2}\mathrm{i}\phi_a \wedge \overline{\psi}_A \wedge \Gamma^a\psi_A + \mathrm{i}k\overline{\Lambda}_{mA} \wedge V^m \wedge \psi_A + kgF_{ab} \wedge \overline{\psi}_A \wedge \Sigma^{ab}\psi_A \\ & + \mathrm{i}kh\phi_a \wedge \overline{\psi}_A \wedge \Gamma^a\psi_A + kt\varepsilon_{AB}F_{ab} \wedge \overline{\psi}_B \wedge \Sigma^{ab}\psi_A \\ & + \mathrm{i}kz\varepsilon_{AB}\phi_a \wedge \overline{\psi}_B \wedge \Gamma^a\psi_A + \mathrm{i}l\varepsilon_{AB}\overline{\Lambda}_{mA} \wedge V^m \wedge \psi_B \\ & + lg\varepsilon_{AB}F_{ab} \wedge \overline{\psi}_A \wedge \Sigma^{ab}\psi_B + \mathrm{i}lh\varepsilon_{AB}\phi_a \wedge \overline{\psi}_A \wedge \Gamma^a\psi_B \\ & + lt\varepsilon_{AB}\varepsilon_{AC}F_{ab} \wedge \overline{\psi}_B \wedge \Sigma^{ab}\psi_C + \mathrm{i}lz\varepsilon_{AB}\varepsilon_{AC}\phi_a \wedge \overline{\psi}_B \wedge \Gamma^a\psi_C = 0, \end{aligned} \tag{24}$$

where use has been made of the developments for $D\phi^a$ and $D\overline{\Lambda}_A$. As in the preceding case, one does not get any information from the projections over VV and ϕV. We have only to examine the terms containing two ϕ's. By separating those with $\overline{\psi}_A\Gamma_a\psi_A$ and $\overline{\psi}_A\Sigma^{ab}\psi_A$, one gets

$$\tfrac{1}{2}\mathrm{i}\phi_a \wedge \overline{\psi}_A \wedge \Gamma^a\psi_A + \mathrm{i}kh\phi_a \wedge \overline{\psi}_A \wedge \Gamma^a\psi_A + \mathrm{i}lz\varepsilon_{AB}\phi_a \wedge \overline{\psi}_B \wedge \Gamma^a\psi_A = 0, \tag{25}$$

whereby it follows that:

$$\tfrac{1}{2} + kh + lz = 0 \tag{26}$$

and

$$kt\varepsilon_{AB}F_{ab} \wedge \overline{\psi}_B \wedge \Sigma^{ab}\psi_A + lg\varepsilon_{AB}F_{ab} \wedge \overline{\psi}_A \wedge \Sigma^{ab}\psi_B = 0; \tag{27}$$

and then a new equation for the parameters

$$kt - g = 0. \tag{28}$$

By following a reasoning analogous to the previous two which we have presented, we develop the Bianchi identities for DA and $D\lambda_A$. We can thereby find the following equations for the parameters:

$$2fk - d = 0, \tag{29}$$
$$2fl - d = 0, \tag{30}$$
$$m = c \quad \text{and} \quad p = d, \tag{31}$$
$$ch + f + dz = 0, \tag{32}$$
$$1 + cg + dt = 0, \tag{33}$$
$$dh - cz = 0, \tag{34}$$
$$dg - ct = 0 \tag{35}$$

obtained from the identity $DDA = 0$ and

$$4 + 3dt + 2z + 3cg + 2kh = 0, \tag{36}$$
$$3ct + 2kz - 3dg - 2lk = 0, \tag{37}$$
$$cg - 2kh = 0, \tag{38}$$
$$dg - 2lh = 0, \tag{39}$$
$$ct - 2kz = 0, \tag{40}$$
$$dt - 2lz = 0 \tag{41}$$

obtained from the identity $DD\lambda_A = 0$.

The rheonomy requirement suggests that $c = f$, so we make both of them equal to one.

The set of equations for the parameters has then been fulfilled. By solving the system (29)–(41) we will find that:

$$\begin{aligned} &n = 1, && k = \tfrac{1}{2}, \\ &p = d = 1, && l = \tfrac{1}{2}, \\ &t = z = -\tfrac{1}{2}, && g = h = -\tfrac{1}{2}. \end{aligned} \tag{42}$$

There is no incompatibility among Eqs. (21)–(41). This is a signal that the hypothesis made for the curvatures is acceptable: a more restrictive hypothesis could imply the non-solvability of the system.

3. Conclusions

The results achieved in this work can be summarized as follows:

(A) The determination, through the analysis of the Bianchi identities, of a compatible system of equations for the parameters of the theory indicates that the hypothesis made for them is acceptable.

(B) Through the explicit determination of the curvatures by means of the Bianchi identities we shall be able to build up, in a future report, the geometrical action which can be readily coupled to the five-dimensional supergravity.

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